DOE/ER/53223-17
MF-111

Courant Institute of
Mathematical Sciences

Magneto-Fluid Dynamics Division

Self-Excitation of the Ballooning Tearing Mode and the Nature of Anomalous Transport in Toroidal Magnetic Confinement Systems

L. E. Zakharov and K. S. Riedel

NYU MF- 111 E
Zakharov, L
Self-excitation of the
ballooning tearing mode
and the nature of... c.1

U.S. Department of Energy

Plasma Physics
August 1986

NEW YORK UNIVERSITY

UNCLASSIFIED

New York University

Courant Institute of Mathematical Sciences

Magneto-Fluid Dynamics Division

MF-111 DOE/ER/53223-17

Self-Excitation of the Ballooning Tearing Mode and the Nature of Anomalous Transport in Toroidal Magnetic Confinement Systems

L.E. Zakharov and K.S. Riedel

U.S. Department of Energy

Grant No. DE-FG02-86ER53223

August 1986

UNCLASSIFIED

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

Printed in U.S.A.

Available from

National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road
Springfield, VA 22161

SELF-EXCITATION OF THE BALLOONING TEARING MODE AND THE NATURE OF ANOMALOUS TRANSPORT IN TOROIDAL MAGNETIC CONFINEMENT SYSTEMS

L.E. Zakharov

Kurchatov Institute of Atomic Energy

U.S.S.R.

and

K.S. Riedel

Courant Institute of Mathematical Sciences

New York University

New York, New York 10012

ABSTRACT

The self-excitation of the ballooning tearing mode in toroidal systems with strong magnetic fields is discovered. This instability is driven by the pressure gradient at small islands and arises from the coupling of poloidal harmonics. This instability is always unstable with respect to perturbations of finite size. In our opinion, it is the main mechanism for plasma turbulence and therefore defines the energy confinement time in tokamaks and stellarators.

In every contemporary auxiliary heating experiment in tokamaks, the loss of energy is anomalously large independent of the level and method of heating. Although the invariance of the loss mechanism is indicative of the magnetohydrodynamic nature of the transport, the physical mechanism for this loss has, until now, not been explained. We will show that in a high temperature toroidal plasma, in which the condition $\vec{B}\cdot\nabla p$ is well satisfied, a finite size MHD perturbation of the ballooning tearing mode type is always unstable and is driven by the local pressure gradient near a small magnetic island. The formation of magnetic islands arising from the ballooning tearing mode, in our opinion, is the dominant cause of anomalous transport in toroidal systems.

To investigate the instability of the tearing mode, we examine the simple, widely-used model for tokamak configurations[1,2] when the metric tensor is given by $g_{11} = 1$, $g_{12} = 0$, $g_{22} = a^2$ and $g_{33} = (R-a\cos\theta)^2$. In accordance with the theory of nearest equilibrum,[2] we will calculate the saturated state of the instability by examining the stability of the perturbed equilibrium with an island of finite width, w. The dominant effect of the perturbation is the flattening of the pressure profile within a distance w of the resonance surface. This is consistent with the numerical simulation in [4]. The flattening of the pressure results from the equilibrium condition $\vec{B}\cdot\nabla p = 0$ and is valid whenever the island formation and growth is slow in comparison to the propagation of sound waves.

For simplicity, we model the effect of the islands by setting $p'(a) = 0$ for $|a-a_m| \leq w$. Thus our actual calculation reduces to the determination of the stability of the step function profile of p' shown

in Fig. 1. Saturation corresponds to marginal stability of the modified profile. Since the rotational transform is approximately zero in the island, the island induces a spatial variation in the shear as well. Near the separatrix the flux averaged shear decreases from a small constant value, S_o to order S_oW. The spatial variation of the shear S(x) enters only in higher order.

We introduce the dimensionless parameter $x = \frac{m(a-a_m)}{a_m}$ where $nq(a_m) = m$ and examine a trial function ξ of the form ϵ

$$\xi(x,\theta,\zeta) = e^{i(m\theta-n\zeta)} \sum_{\ell} F_\ell(x)e^{i\ell\theta} \tag{1}$$

The standard expression for the potential energy of a magnetohydrodynamic plasma is

$$\delta W = \sum_{\ell} \int_{-\infty}^{\infty} dx\Big[[(\ell-S(x)x)F_\ell]'^2 + (\ell-S(x)x)^2F_\ell^2 + (\frac{\alpha^2}{2} + \alpha U)F_\ell^2 + \frac{\alpha}{2} F_\ell'(F_{\ell+1}-F_{\ell-1}) - \alpha F_\ell F_{\ell+1}\Big] \tag{2}$$

where S(x) is the shear, U is the magnetic well, $U = \frac{a}{R}\left(1 - \frac{1}{q^2}\right)$ and $\alpha(x) = -8\pi p' Rq^2/B_S^2$ is shown in Fig. 1.

We assume, for simplicity that $S,\alpha << 1$ and that the instability is localized near a single resonance surface, a_o. This last assumption implies that destabilizing effects of order $\alpha \exp(-1/S)$ will be neglected. We restrict to the case when the probe function of the energy principle consist of only three poloidal harmonics, F_o, F_1, F_{-1}. The Euler equations for (2) are

$$\frac{d}{dx} x^2 S(x)^2 \frac{d}{dx} F_o - S^2 x^2 F_o = \left(\frac{\alpha^2}{2} + \alpha U\right) F_o - \alpha(P+D') - \frac{\alpha'(x) D}{2} \qquad (3a)$$

$$P'' - P = -\frac{\alpha F_o}{2} \qquad (3b)$$

$$D'' - D = \frac{\alpha' F_o}{4} + \frac{\alpha(x)}{2} F_o' \qquad (3c)$$

where $P \equiv \frac{F_{+1}+F_{-1}}{2}$ and $D \equiv \frac{F_{+1}-F_{-1}}{2}$. Equations (3b) and (3c) have solutions given by

$$\begin{aligned} P &= \frac{1}{4} \int_{-\infty}^{\infty} \alpha(t) F_o(t) e^{-|x-t|} dt \\ D &= \frac{1}{4} \int_{-\infty}^{\infty} \left[\alpha(t) \Lambda(x-t) + \frac{\alpha'(t)}{2}\right] F_o(t) e^{-|x-t|} \end{aligned} \qquad (4)$$

where $\Lambda(x) \equiv 1$ when $x > 0$ and $\Lambda(x) \equiv -1$ for $x < 0$. After applying these expressions for P and D the variational principle becomes

$$\begin{aligned} \delta W = &\int_{-\infty}^{\infty} S(x)^2 \left[(xF_o)'^2 + (xf_o)^2 + \alpha U F_o^2\right] \\ &- \frac{1}{4} \int_{-\infty}^{\infty} \alpha'(x) F_o(x) dx \int_{-\infty}^{\infty} \left[\alpha(t) \Lambda(x-t) + \frac{\alpha'(t)}{4}\right] F_o(t) e^{-|x-t|} \end{aligned} \qquad (5)$$

We insert the simple trial function $F_o(x) = e^{-|x|}/x$ corresponding to the tearing mode. For the step function model of the pressure distribution, δW simplifes to

$$\delta W = \int_{-\infty}^{\infty} S(x)^2 \, (xF_o)'^2 + x^2F_o^2] + 2\int_{-\infty}^{\infty} \alpha U F_o^2 dx + \alpha^2 F(w) \, \text{ch } w \, [\int_w^{\infty} F_o(t)e^{-t} - \frac{1}{4} e^{-w}F_o(w)]$$

$$\leq 2S^2 + \frac{2\alpha U e^{-2w}}{w} + \frac{\alpha^2}{w} [-.577 - \ell n \, 2w - 1/(4w)] \, (\frac{1 + e^{-2w}}{2})e^{-2w} \quad (6)$$

Clearly for any values of the parameters, $.577 + \ell n \, 2w + 1/(4w) = 0$ determines a bound on the width of the island. This bound has the numerical value $W = .3$. In these units, the distance between resonant surfaces is $1/S$. Not only are all nested flux surface equilibria unstable to finite size perturbation, but the combination of many different poloidal and toroidal harmonics will result in island overlap. The existence of equilibria with a finite number of islands is therefore doubtful. Thus the plasma must consist of large regions of ergodic field lines or constantly move between a statistical ensemble of equilibrium states.

In the limit as W tends to zero, the energy is clearly negative and the magnetic well canot stabilize this instability. The width of the island is determined by the marginal stability of the pressure distribution, i.e., $\delta W = 0$. The island width is displayed as a function of α for fixed U and S in Fig. 2. When α is much less than $\sqrt{8}$ S the island is small and satisfies

$$W \leq \frac{\alpha/4}{U + \sqrt{U^2 + S^2/2}} \quad (7)$$

The substitution of $a_m W/m$ for W converts this expression to a dimensional width.

Surprisingly, the quasilinear calculation has shown the appearance of a new linear instability on the modified profile. Furthermore the limit as W tends to zero is discontinuous. Any finite size island in a toroidal system will cause this instability even when the plasma is far from marginal instability of linear theory for nested flux surfaces. We call this phenomenon plasma self-excitation.

The instability is caused by the large second derivative of the pressure near the separatrix, $p''(a_{sep})$, driving the satellite harmonics. In the traditional tearing mode in a cylinder the larger derivatives of $j(a)$ and $p'(a)$ are destabilizing on the inside of the island and stabilizing outside. Thus these effects cancel to leading order. However in a toroidal geometry, the satellite harmonics are destabilized on both sides of the island. This effect is independent of the assumption that m,n are much greater than one.

The calculation reveals the presence of a lower energy state with islands and δW is the release of potential energy. Thus our δW can be viewed as the generalization of the resistive energy principle of Furth[2] to a toroidal geometry with islands. δW can also be interpreted as a scaled Δ' for tearing ballooning modes with islands.

Implicit in our calculation is the assumption that the island is larger than the resistive layer and thus we expect the island to evolve in the Rutherford regime. For thermonuclear plasma, the resistive layer is extremely small and so effectively there is no nonlinear threshold for the size of the perturbation. Since for all values of beta, there are equilibrum states with islands at lower potential energy, the assumption of nested flux surfaces is always questionable. Furthermore the overlap of islands occurs at much lower levels of beta

than previously believed. Therefore we hypothesize that the destruction of flux surfaces resulting from the tearing ballooning mode is a principle cause of anomalous transport.

Our stability criteria, $\Delta W > 0$, determines whether the potential energy outside the resistive layer drives the instability. Thus our results are independent of the model of dissipation used in the resistive layer.

ACKNOWLEDGMENTS

This work was done at the Kurchatov Institute of Atomic Energy as part of the U.S.-U.S.S.R. exchange program in fusion energy. Dr. Riedel wishes to thank Drs. V.D. Shafranov, V.D. Pustovitov and S.B. Semenov for their hospitality. Dr. Riedel also thanks Prof. Freidberg and Prof. Bers for a stimulating discussion.

This work was supported by U.S. Department of Energy Grant No. DE-FG02-86ER53223.

APPENDIX: TWISTING MODE

We will now calculate the nonlinear behavior of the odd twisting mode[1] in physical space. We exactly reproduce the results of Zakharov, Semenov and Riedel, however our present calculation is substantially simpler and more intuitive than the previous analysis in Fourier space. We again base our analysis on the theory of nearest equilibrium and therefore analyze the stability of the nested flux surface equilibrium with a pressure distribution given in Fig. 2.

Our starting point is the energy function given in Eq. (2). The analysis of the twisting mode requires the determination of the next order terms and the retention of the second satellite harmonic. The general Euler equations are

$$\frac{d}{dx}(m-Sx)^2\frac{d}{dx}F_m - (m-Sx)^2F_m$$

$$= f_m \equiv \left(\frac{\alpha^2}{2} + \alpha U\right)F_m - \frac{\alpha}{2}\left(F_{m+1}+F_{m-1}\right) + \frac{\alpha}{2}\left(\frac{dF_{m-1}}{dx} - \frac{dF_{m+1}}{dx}\right) + \frac{d\alpha}{dx}\left(\frac{F_{m-1}-F_{m+1}}{4}\right) \quad (A1)$$

This equation can be solved with a Green's function to yield

$$F_m(x) = -\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{-|x-t|}f_m}{(m-Sx)(m-St)} \quad (A2)$$

The following identities will be useful

$$F_m + \frac{dF_m}{dx} = -\int_x^\infty \frac{e^{x-t} f_m(t)}{(m-Sx)(m-St)} + \frac{(xS(x))' F_m(x)}{(m-Sx)}$$

$$F_m - \frac{dF_m}{dx} = -\int_{-\infty}^x \frac{e^{t-x} f_m(t)}{(m-Sx)(m-St)} - \frac{(xS(x))' F_m(x)}{(m-Sx)} \qquad \text{(A3)}$$

We assume the standard ballooning mode ordering, that α is small, the shear S and the magnetic well W are order α^2 and that the width of the island is also small. We neglect the destabilization due to the resonance of the satellite harmonics at their corresponding resonant surfaces. This assumption of the localization of the eigenfunction around the resonance surface of the fundamental harmonic is completely equivalent to the averaging of the equation over small shear in the Connor-Hastie-Taylor formalism. The inclusion of other resonance surfaces in this analysis would result in the presence of the standard destabilizing term of order $\alpha \exp(-1/S)$ in Eq. (A7).

With these assumptions, $F_{\pm 1}$ are order α and $F_{\pm 2}$ are order α^2. Thus $f_{\pm 2}$ can be approximated by

$$f_{\pm 2} \simeq -\frac{\alpha(x) F_{\pm 1}}{2} \pm \frac{d}{dx}\left(\frac{\alpha F_{\pm 1}}{2}\right) \mp \frac{d\alpha}{dx}\frac{F_{\pm 1}}{4}$$

For convenience we divide $F_{\pm 1}$ into the sum of $F^o_{\pm 1}$ and $F^1_{\pm 1}$ where $F^o_{\pm 1}$ is forced by the fundamental harmonic F_o. Thus

$$f^{o}_{\pm 1} \equiv -\frac{\alpha F_o}{2} \pm \frac{d}{dx}\left(\frac{\alpha F_o}{2}\right) \mp \frac{d\alpha}{dx}\frac{F_o}{4}$$

$$f^{1}_{\pm 1} \equiv \frac{\alpha^2 F_{\pm 1}}{2} - \frac{\alpha}{2} F_{\pm 2} \mp \frac{d}{dx}\left(\frac{\alpha F_{\pm 2}}{2}\right) \pm \frac{d\alpha}{dx}\frac{F_{\pm 2}}{4}$$

Integrating Eq. (A2) by parts for F^{o}_{+1} and F^{o}_{-1} yields

$$\begin{aligned} F^{o}_{+1} &\simeq \frac{1}{2}\int_{-\infty}^{x} \alpha(t)F_o e^{t-x} + \frac{1}{8}\int_{-\infty}^{\infty} \frac{d\alpha}{dt} F_o(t) e^{-|x-t|} \\ F^{o}_{-1} &\simeq \frac{1}{2}\int_{x}^{\infty} \alpha(t)F_o e^{x-t} - \frac{1}{8}\int_{-\infty}^{\infty} \frac{d\alpha}{dt} F_o(t) e^{-|x-t|} \end{aligned} \qquad \text{(A4)}$$

Furthermore, we can integrate (A3) by parts to find

$$\alpha(x)\left[F_{\pm 2} \pm \frac{dF_{\pm 2}}{dx}\right] \simeq \frac{\alpha^2(x)F_{\pm 1}}{8} + O(\alpha^2 W)$$

We use this expression in (A3) to determine the desired combinations of $F_{\pm 1}$

$$\alpha(x)\left[F_{+1} + \frac{dF_{+1}}{dx}\right] = \frac{\alpha(x)}{2}\int_{x}^{\infty} \frac{e^{x-t}\left[\alpha F_o - \frac{d(\alpha F_o)}{dt} + \frac{d\alpha}{dt}F_o/2 - \frac{7}{8}\alpha^2 F_1\right]}{(1-Sx)(1-St)} + \alpha\left(xS(x)\right)' F_{+1}(x)$$

$$\simeq \frac{\alpha^2(x)F_o}{2} + \frac{\alpha(x)}{4}\int_{x}^{\infty}\left[\left(\frac{d\alpha}{dt} + \left(tS(t)\right)'S\right)F_o - \frac{7}{4}\alpha^2 F_1(t)\right]e^{x-t} + \alpha\left(xS(x)\right)' F_{+1}(x)$$

$$\alpha(x)\left[F_{-1} - \frac{dF_{-1}}{dx}\right] = \frac{\alpha(x)}{2}\int_{-\infty}^{x} \frac{e^{t-x}\left[\alpha F_o + \frac{d(\alpha F_o)}{dt} - \frac{d\alpha}{dt}F_o/2 - \frac{7}{8}\alpha^2 F_1(t)\right]}{(1+Sx)(1+St)} + \alpha\left(xS(x)\right)' F_{-1}(x)$$

$$\simeq \frac{\alpha^2(x)F_o}{2} + \frac{\alpha(x)}{4}\int_{-\infty}^{x}\left[\left(\frac{d\alpha}{dt} + 2\left(xS(x)\right)'\right)F_o - \frac{7}{4}\alpha^2 F_{-1}(t)\right]e^{t-x} + \alpha\left(xS(x)\right)' F_{-1}(x)$$

These expressions can be simplifed by using the approximate identity

$$\int_{-\infty}^{x} e^{t-x}a\alpha^2(t)F_{-1}(t) \simeq \frac{1}{2}\int_{-\infty}^{x} e^{t-x}\alpha^2(t)\int_{x}^{\infty} e^{t-u}\alpha(u)F_o(u)du$$

$$\simeq \frac{1}{2}\int_{-\infty}^{x} e^{t-x}\,\bar{\alpha}^2\int_{t}^{\infty} e^{t-u}d(u)F_o(u)$$

$$\simeq \frac{1}{4}\left[\int_{-\infty}^{x} e^{t-x}\alpha^3(t)F_o(t) + \alpha^2(x)\int_{x}^{\infty} e^{x-u}\alpha(u)F_o(u)\right]$$

where $\bar{\alpha}$ is average value of $\alpha(x)$.

We are now able to eliminate the satellite harmonics and re-express f_o as

$$f_o = \alpha U F_o - \frac{\alpha(x)}{8}\int_{\infty}^{\infty}\frac{d\alpha}{dt}F_o(t)\Lambda(t-x)^{-|x-t|} + \frac{\alpha}{2}\left(\frac{7\alpha^2}{32} - \left(xS(x)\right)'\right)\int_{-\infty}^{\infty} e^{-|x-t|}\alpha F_o$$

$$+ \frac{1}{8}\left(\frac{d\alpha}{dx}\right)\left[\int_{-\infty}^{\infty}\alpha(t)\Lambda(t-x)F_o)e^{-|t-x|} - \frac{1}{2}\int_{-\infty}^{\infty} e^{-|t-z|}F_o\right] \qquad \text{(A5)}$$

where Λ is defined in Eq. (4). If the width of the island is set equal to zero our equation is the Fourier transform of Strauss's equation. Until now the calculation has been valid for an arbitrary distribution of pressure $\alpha(x)$ and shear $S(x)$ in the island region. We now restrict to the step function distribution for $\alpha(x)$.

The contribution of the island on the potential energy is

$$\delta W_{isl} = \alpha^2 F(w) W\left(\mu - \frac{F_o(w)e^{-W}}{4}\right) \qquad (A6)$$

where

$$\mu \equiv \int_w^\infty e^{-t} F_o(t)$$

and

$$W(w) \equiv \frac{e^W - e^{-W}}{2}$$

Since $F_o(w)$ has a logarithmic singularity this energy will be negative and thus the islands have a destabilizing effect on the twisting mode as well. Thus twisting modes also exhibit self-excitation, however the destabilizing term scales as the width w while the corresponding term in the tearing mode scales as 1/w. Thus the presence of islands destabilize the twisting mode only near the marginal stability boundary. We determine the stability criteria for twisting modes by evaluating δW with the trial function $F_o(x) = K_o(x)$, the modified Bessel function.

This trial function is the exact solution for marginally stable twisting modes when the magnetic well, U, is set to zero and there are no islands.[1] Evaluation of (A6) gives

$$\delta W = (1-\lambda^2)\frac{\pi^2 S^2}{4} + \frac{\alpha U \pi^2}{2} + \alpha^2 w K_o(w)\left[1 - \frac{K_o(w)}{4}\right] \qquad (A7)$$

where $\lambda^2 \equiv \frac{\alpha^2}{S}\left(1 - \frac{7}{32}\frac{\alpha^2}{S}\right)$ and $\lambda = 1$ corresponds to marginal stability

when U = 0. In an initially marginally stable equilibrium the islands will grow to a width

$$w = .0205$$

In Strauss's model the region of linear instability is $1 - \frac{\sqrt{2}}{4} \leq \frac{7}{16} - \frac{\alpha^2}{S} \leq 1 + \frac{\sqrt{2}}{4}$ and the center of this region is $S/\alpha^2 = 7/16$. The value $\alpha^2/S = 16/7$ not only has the maximal value of the linear increment but also the largest island width

$$w = .02 + .015S$$

Finally, the potential of the magnetic island is at a minimum when w = .006 and gives a nonlinear stability criteria

$$\delta W = (1-\lambda^2)\,\frac{\pi^2 S^2}{4} + \frac{\alpha U \pi^2}{2} - .001\alpha^2$$

If ΔW is positive, the potential energy outside the resistive layer is stabilizing. Thus when the twisting mode is in the region of $\Delta W > 0$ and $\Delta' > 0$, the mode must be driven inside the resistive layer. Hence an instability of this type will saturate with an amplitude inside the resistive layer. Furthermore the stability of this region depends on the model of dissipation chosen. Finally, when the twisting mode has $\Delta W > 0$ and $\Delta' > 0$, the mode tends to different values of b_r on opposite sides of the mode rational surface. Thus the eigenfunction must be considerably altered by resistivity to produce this mode.

We note that, due to the crudeness of our model of the pressure distribution, these results are only qualitatively but not quantitatively correct.

REFERENCES

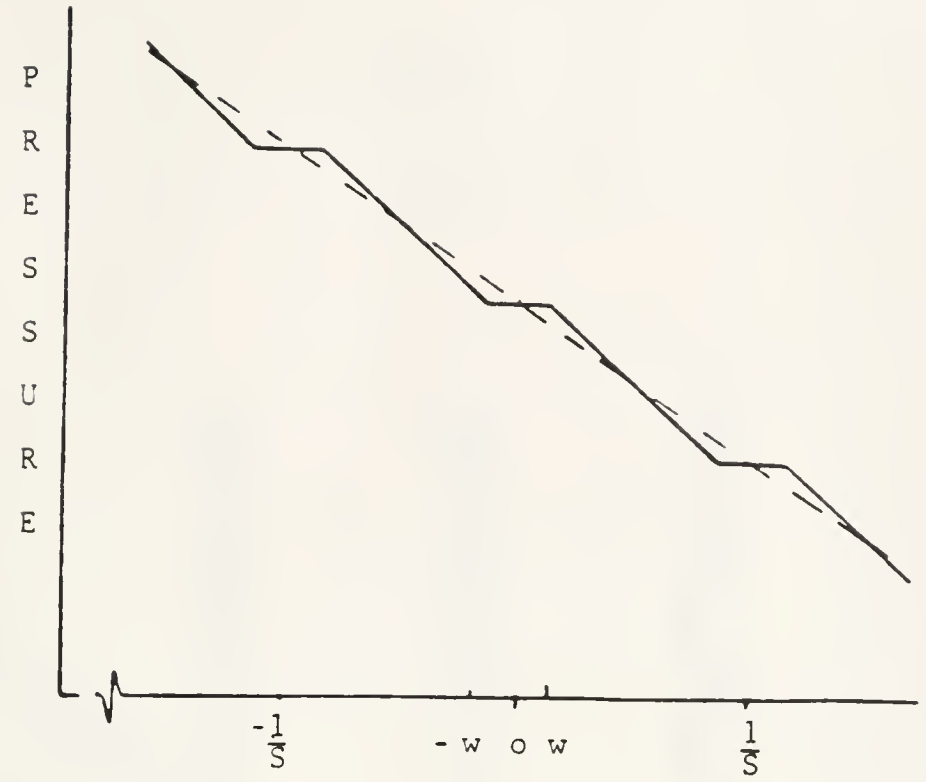
1. H.R. Strauss, Phys. Fluids 24, 2004, 1987.

2. L.E. Zakharov, S.B. Semenov and K.S. Riedel, Letters in JETP, Vol. 44, 25 (1986).

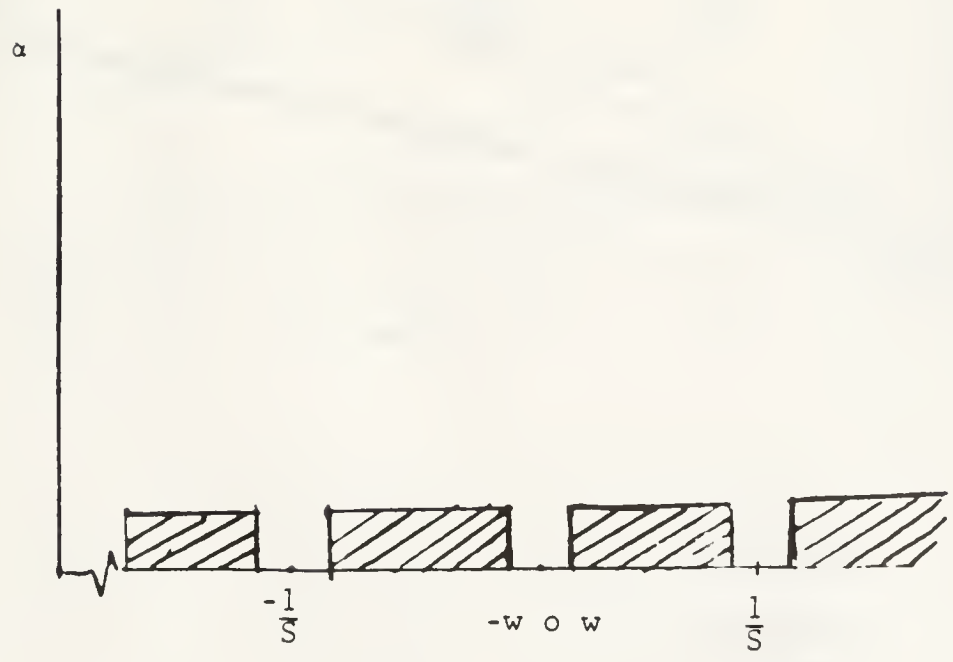
3. H.P. Furth, P.H. Rutherford and H. Selberg, Phys. Fluids 16, 1054 (1973).

4. B.A. Carreras, P.H. Diamond, et al., Phys. Rev. Lett., Vol. 50, No. 7 (1983), p. 503.

FIGURE 1



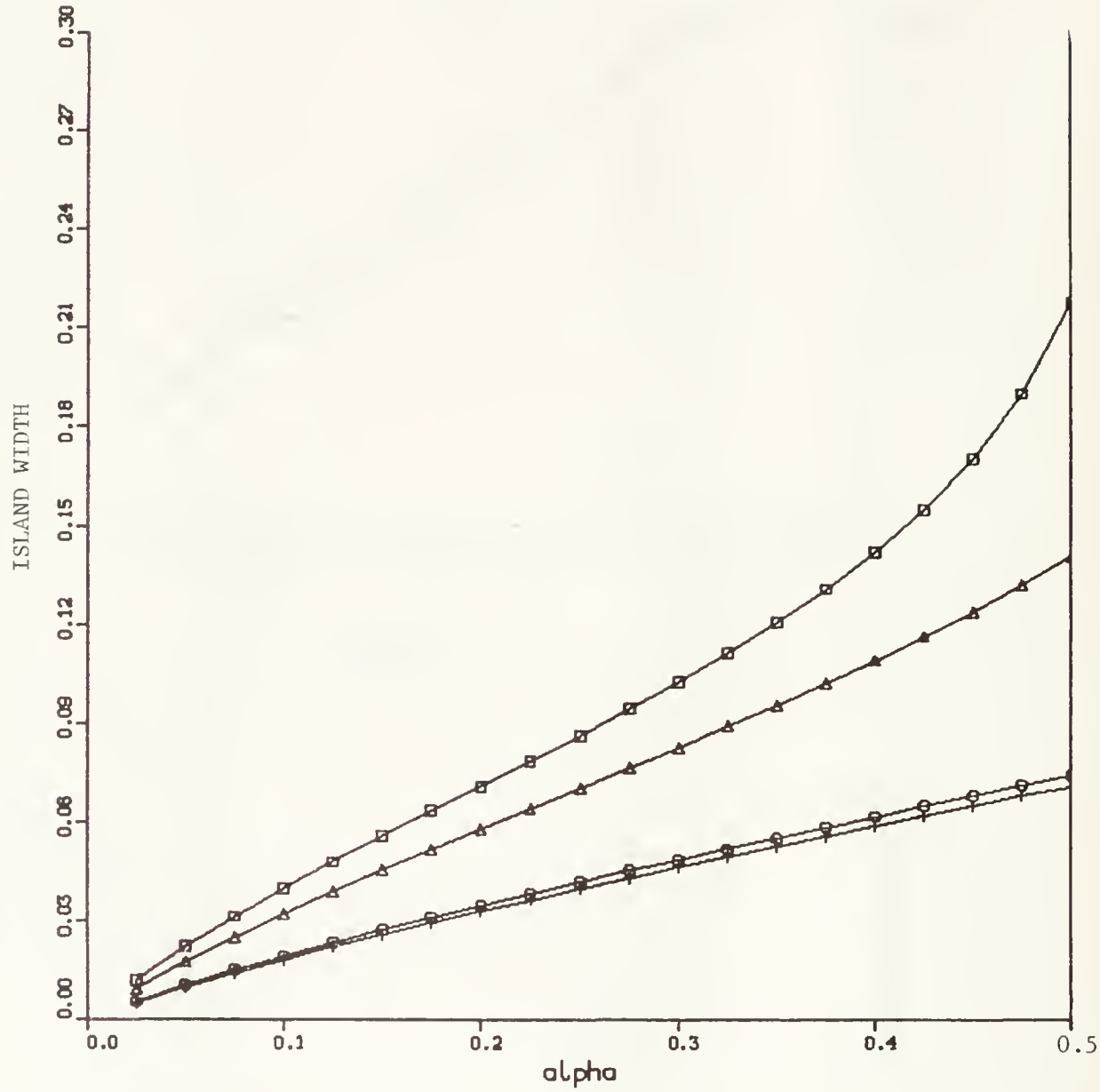
$$x = \frac{a - a_m}{m}$$



Pressure profile and step function model for

$$\alpha(x) \equiv \frac{-8\pi' Rq^2}{B_T}.$$

FIGURE 2



Shear = 0.5 Well = 0.5

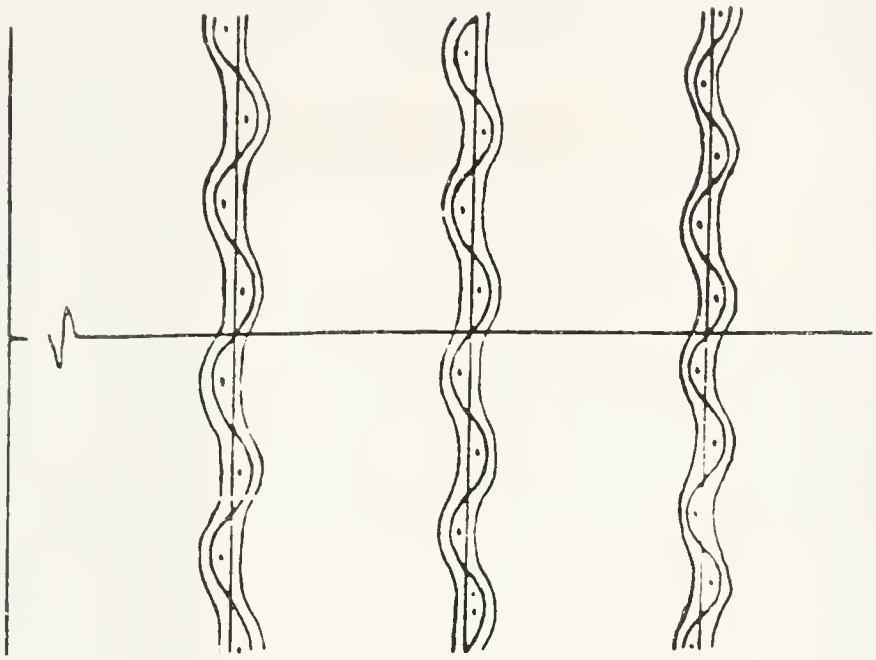
Shear = 0.2 Well = 0.5

Shear = 0.5 Well = 0.2

Shear = 0.2 Well = 0.2

Saturated island width w as a function of α for fixed shear S and well depth u.

FIGURE 3



Twisting Ballooning Mode Islands

DATE DUE

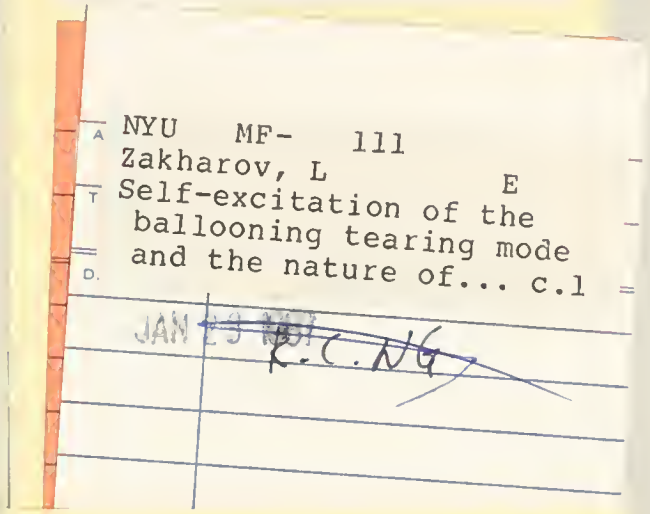
DEC 1 6 1986

JAN 2 3 1987

GAYLORD

PRINTED IN U.S.A.

NYU MF- 111
Zakharov, L E
Self-excitation of the
ballooning tearing mode
and the nature of... c.1


NYU MF- 111
Zakharov, L E
Self-excitation of the
ballooning tearing mode
and the nature of... c.1
JAN 23 R.C.N6

LIBRARY
N.Y.U. Courant Institute of
Mathematical Sciences
251 Mercer St.
New York, N. Y. 10012

LIBRARY
CIMS